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On Gekeler's Conjecture for Function Fields

Bruno Anglès[1]

Laboratoire de Mathématiques Emile Picard, C.N.R.S. U.M.R. 5580, U.F.R. M.I.G., Université Paul-Sabatier, 118, Route de Narbonne, 31062 Toulouse Cedex 4, France
E-mail: angles@picard.ups-tlse.fr

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Let P be a monic irreducible polynomial in $\mathbb{F}_q[T]$ such that $d = \deg P$ is even. We have obtained (B. Anglès, 1999, *J. Number Theory* **79**, 258–283), when q is odd,

only if the Bernoulli–Carlitz number $B((q^d-1)/2)$ is divisible by P. This result is a special case of Gekeler's conjecture. In these notes, we give a class number congruence for the ideal class number of any totally real subfield F of the Pth cyclotomic function field (Theorem 2). As expected, this formula involves the Bernoulli–Carlitz numbers. It also appears in this formula which we call the regulator modulo P of F: R_F. In the case where $F = \mathbb{F}_q(T, \sqrt{P})$, then $R_F \not\equiv 0 \pmod{P}$. Unfortunately, this is not the case for general F. In Section 3, we show that if $R_F \not\equiv 0 \pmod{P}$, then Gekeler's conjecture is true for the field F (Theorem 4).

1. GEKELER'S CONJECTURE

In this section, we give some notations and we state Gekeler's conjecture.

Let $\mathbb{F}_q$ be a finite field having q elements and let p be the characteristic of $\mathbb{F}_q$. Let T be an indeterminate over $\mathbb{F}_q$, set $Z = \mathbb{F}_q[T]$, and $Q = \mathbb{F}_q(T)$. Let P be an irreducible and unitary element in Z and set $d = \deg P$. Let Q_P be a P-adic completion of Q and let Q_P^{ac} be a fixed algebraic closure of Q_P. We view Q as contained in Q_P. Let Z_P be the valuation ring of Q_P, then there exists an unique finite field $\mathbb{F}_P$ contained in Z_P such that $\mathbb{F}_P \simeq Z/PZ$.

Let $Z_P \to \mathrm{End}_{Z_P} \mathbb{G}_a$, $a \mapsto [a]_C$, be the P-adic Carlitz module; i.e., it is a ring homomorphism such that $[T]_C = X^q + TX$ and for all $a \in Z_P$, $[a]_C \equiv aX \pmod{\deg 2}$. Set

$$\Lambda_P = \{z \in Q_P^{\mathrm{ac}},\ [P]_C(z) = 0\}.$$

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Set $K = Q_P(\Lambda_P)$. We fix $\lambda_P \in \Lambda_P$, $\lambda_P \neq 0$. The Pth cyclotomic function field is equal to

$$Q(\lambda_P) \subset K.$$

Let F/Q be a totally real subextension of $Q(\lambda_p)/Q$, i.e., $1/T$ splits completely in F. We denote the integral closure of Z in F by Q_F, the group of units of O_F by E_F. Recall that the group of cyclotomic units of F, Cyc_F, is the subgroup of E_F generated by $\mathbb{F}_q^*$ and by

$$N_{Q(\lambda_P)^+/F}\left(\frac{\sigma(\lambda_P)}{\lambda_P}\right), \qquad \sigma \in \mathrm{Gal}(Q(\lambda_P)/Q),$$

where $Q(\lambda_P)^+ = Q(\lambda_P^{q-1})$ is the maximal totally real subextension of $Q(\lambda_P)/Q$. Set $\ell = [F : Q]$, $\ell \geqslant 2$. Set $G_F = \mathrm{Gal}(F/Q)$ and $G = \mathrm{Gal}(Q(\lambda_P)/Q)$. Let Cl_F be the p-Sylow subgroup of the ideal class group of O_F.

Let $\zeta - q^d - 1$ be a primitive $(q^d - 1)$st root of 1 and let W be the valuation ring of $\mathbb{Q}_p(\zeta_{q^d-1})$. Let *Frob* be the Frobenius of $\mathbb{Q}_p(\zeta_{q^d-1})/\mathbb{Q}_p$, i.e., *Frob* is the unique element of $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{q^d-1})/\mathbb{Q}_p)$ such that

$$Frob(\zeta_{q^d-1}) = \zeta_{q^d-1}^p.$$

Note that *Frob* acts on the $W[G_F]$-module,

$$Cl_F \otimes_{\mathbb{Z}_p} W.$$

Let $\omega_P \colon G \to W^*$ be the Teichmüller character (see [5, Sect. 1]) and set $\chi_P = \omega_P^{(q^d-1)/\ell}$. For $j = 1, ..., \ell$, we set

$$e_j = \frac{1}{\#(G_F)} \sum_{\tau \in G_F} \chi_P^j(\tau)\, \tau^{-1} \in W[G_F].$$

We set

$$Cl_F(j) = (Cl_F \otimes_{\mathbb{Z}_p} W)\, e_j.$$

Then

$$Cl_F \otimes_{\mathbb{Z}_p} W = \bigoplus_{j=1}^{\ell-1} Cl_F(j).$$

Now, we define an equivalence relation on the set $\{1, ..., \ell - 1\}$, we write $j' \sim_F j$ if and only if there exists an integer $n \geqslant 0$ such that $j' \equiv p^n j \pmod{\ell}$. For $j \in \{1, ..., \ell - 1\}$, we set

$$[j]_F = \{j' \in \{1, ..., \ell - 1\},\ j' \sim_F j\}.$$

Note that $[j]_F = \{j\}$ for all $j \in \{1, ..., \ell - 1\}$, if and only if $p \equiv 1 \pmod{\ell}$. Furthermore, if $j' \equiv pj \pmod{\ell}$, we have

$$Frob\, e_j = e_{j'}.$$

Therefore, for $j_1, j_2 \in [j]_F$, we have

$$Cl_F(j_1) \simeq Cl_F(j_2). \tag{1}$$

Therefore we can define $Cl_F([j]_F)$ for $j \in \{1, ..., \ell - 1\}$. We have the following theorem due to D. Goss and S. Okada:

THEOREM 1. *We have*

$$Cl_F([j]_F) \neq (0) \Rightarrow \forall j' \in [j]_F, \qquad B\left(j' \frac{q^d - 1}{\ell}\right) \equiv 0 \pmod{P},$$

where $B(k)$ is the kth Bernoulli–Carlitz number.

Proof. See [5, Theorem 2.18]. ∎

In [4], E.-U. Gekeler has made the following conjecture, which is quite similar to Ribet's Theorem:

Let F/Q be a totally real subextension of $Q(\lambda_P)$, $[F:Q] = \ell \geqslant 2$. Let $j \in \{1, ..., \ell - 1\}$, then $Cl_F([j]_F) \neq (0)$ if and only if for all j' in $[j]_F$, $B(j'(q^d - 1)/\ell) \equiv 0 \pmod{P}$.

In [1], this conjecture has been proved in the case where q is odd, $d \equiv 0 \pmod{2}$ and $F = Q(\sqrt{P})$. The general case is still an open problem.

2. A CLASS NUMBER CONGRUENCE

In this section, we give a class number congruence for the ideal class number of a totally real subfield of the Pth cyclotomic function field. This formula is very similar to that obtained by T. Metänkylä (see [7]).

Recall that $K = Q_P(\lambda_P)$, we denote the maximal ideal of the valuation ring of K by $\mathfrak{p}_K$. Let U_K be the group of units of K, we set, for $n \geqslant 1$, $U_K^{(n)} = 1 + \mathfrak{p}_K^n$. We recall briefly the definition of the Carlitz–Kummer morphisms (see [1, Sect. 2]).

Let $L(X) = X^{q^d} + PX \in Z_P[X]$ be the basic Lubin–Tate polynomial. Then L induces a ring homomorphism: $Z_P \to \mathrm{End}_{Z_P} \mathbb{G}_a$, $a \mapsto [a]_L$. Set

$$\mathrm{Log}_L(X) = \lim_{n \geqslant 1} \frac{1}{P^n} [P^n]_L \in Q_P[[X]].$$

Let $\mathrm{Exp}_L(X) \in Q_P[[X]]$ be such that

$$\mathrm{Exp}_L(X) \circ \mathrm{Log}_L(X) = \mathrm{Log}_L(X) \circ \mathrm{Exp}_L(X) = X.$$

Set

$$\mathrm{Log}_C(X) = \lim_{n \geqslant 1} \frac{1}{P^n} [P^n]_C \in Q_P[[X]].$$

Then $\mathrm{Log}_C(X)$ is called the Carlitz' logarithm. Let $\mathrm{Exp}_C(X)$ be the inverse for the composition of $\mathrm{Log}_C(X)$. Set $D_0 = 1$, and for $i \geqslant 1$, $D_i = (T^{q^i} - T)\, D_{i-1}^q$, then one can show that

$$\mathrm{Exp}_C(X) = \sum_{i \geqslant 0} \frac{X^{q^i}}{D_i}.$$

Let $k \geqslant 0$ be an integer, write $k = a_0 + a_1 q + \cdots + a_N q^N$, where $0 \leqslant a_i \leqslant q-1$ for $i = 0, ..., N$. Set $\Gamma_k = \prod_{i=0}^N (D_i)^{a_i}$. Then the Bernoulli–Carlitz numbers $B(k)$ are defined by the formula

$$\frac{X}{\mathrm{Exp}_C(X)} = \sum_{k \geqslant 0} \frac{B(k)}{\Gamma_k} X^k.$$

Set

$$g_P(X) = \mathrm{Exp}_L(X) \circ \mathrm{Log}_C(X).$$

Then $g_P(X) \in Z_P[[X]]$ and we set

$$\lambda_L = g_P(\lambda_P).$$

We have $\lambda_L^{q^d - 1} = -P$ and $K = Q_P(\lambda_L)$.

Now, let $u \in U_K$, write $u = h(\lambda_L)$ for some $h(X)$ in $Z_P[[X]]$. Then $h'(\lambda_L)/u$ is well defined modulo $\mathfrak{p}_K^{q^d-2}$. Thus, one can write

$$\frac{h'(\lambda_L)}{h(\lambda_L)} \equiv \sum_{k=1}^{q^d-2} \varphi_k(u)\, \lambda_L^{k-1} \qquad (\mathrm{mod}\ \mathfrak{p}_K^{q^d-2}), \tag{2}$$

where $\varphi_k(u) \in \mathbb{F}_P$ for $k = 1, ..., q^d - 2$. Then $\varphi_k\colon U_k \to \mathbb{F}_P$ is a morphism of groups which is called the Carlitz–Kummer morphism of degree k. By [1, Proposition 2.4], we have

$$\bigcap_{1 \leqslant k \leqslant q^d - 2} \mathrm{Ker}\, \varphi_k = (U_K)^p\, U_K^{(q^d-1)}. \tag{3}$$

Now, let F be a totally real subfield of $Q(\lambda_P) \subset K$, $[F:Q] = \ell$, $\ell \geqslant 2$. By [3], if h_F is the ideal class number of O_F, we have

$$h_F = (E_F : Cyc_F). \tag{4}$$

Therefore

$$h_F (q-1)^{\ell} = (E_F : (Cyc_F)^{q-1}).$$

Now, let $G \in Z$ be such that G is a primitive root modulo P. For $1 \leqslant i \leqslant q^d - 1$, we set

$$u_i = \frac{[G^i]_C (\lambda_P)}{[G^{i-1}]_C (\lambda_P)}. \tag{5}$$

Let $\gamma \in \mathrm{Gal}(Q(\lambda_P)/Q)$ be such that

$$\gamma(\lambda_P) = [G]_C (\lambda_P).$$

Then γ generates $\mathrm{Gal}(Q(\lambda_P)/Q)$. Set $\tau = \gamma^{\ell}$, then τ generates $\mathrm{Gal}(Q(\lambda_P)/F)$. Now, if $j = i + k\ell$, we have

$$u_j = \tau^{\ell}(u_i).$$

Therefore

$$N_{Q(\lambda_P)/F}(u_i) = N_{Q(\lambda_P)/F}(u_j).$$

For $i = 1, ..., \ell - 1$, we set

$$\eta_i = \prod_{j=0}^{((q^d-1)/\ell)-1} u_{i+j\ell} = N_{Q(\lambda_P)/F}(u_i). \tag{6}$$

It is clear that $\eta_1, ..., \eta_{\ell-1}$ generate $(Cyc_F)^{q-1}$. By [1, Sect. 4], if $k \not\equiv 0 \pmod{(q^d-1)/\ell}$, we have

$$\varphi_k(E_F) = 0.$$

Furthermore, by [1, Sect. 4], for $1 \leqslant k \leqslant q^d - 1$, we have

$$\varphi_k\left(\frac{\lambda_L}{\lambda_P}\right) \equiv \frac{B(k)}{\Gamma_k} \pmod{P}. \tag{7}$$

For $k=1, ..., q^d-2$, $B(k)$ is P-integral. Thus, if $k=j((q^d-1)/\ell)$, $1\leqslant j\leqslant \ell-1$, we get

$$\varphi_k(\eta_i)\equiv\frac{1}{\ell}\,(G^{ik}-G^{(i-1)\,k})\,\frac{B(k)}{\Gamma_k} \qquad (\mathrm{mod}\ P). \tag{8}$$

We consider the following injective homomorphism of groups $\mathscr{L}\colon U_K/((U_K)^p\,U_K^{(q^d-1)})\to\mathfrak{p}_K/\mathfrak{p}_K^{q^d}$ defined by

$$\mathscr{L}(u)\equiv\sum_{k=0}^{q^d-2}\varphi_k(u)\,\lambda_L^k \qquad (\mathrm{mod}\ \mathfrak{p}_K^{q^d}). \tag{9}$$

For $\varepsilon\in E_F$, we get

$$\mathscr{L}(\varepsilon)=\sum_{j=1}^{\ell-1}\varphi_{j((q^d-1)/\ell)}(\varepsilon)\,\lambda_L^{j((q^d-1)/\ell)}. \tag{10}$$

Let $\varepsilon_1, ..., \varepsilon_{\ell-1}$ be a system of fundamental units of E_F. We define the regulator modulo P, R_F, to be

$$R_F\equiv(\det(\varphi_{j((q^d-1)/\ell)}(\varepsilon_i))_{1\leqslant i,\,j\leqslant\ell-1})^2 \qquad (\mathrm{mod}\ P). \tag{11}$$

One can easily show that R_F does not depend on the choice of our system of fundamental units of E_F. Then, by the proof of [7, Theorem 1A], using $\mathscr{L}$ instead of the p-adic logarithm in our case, we get

$$(\det(\varphi_{j((q^d-1)/\ell)}(\eta_i))_{1\leqslant i,\,j\leqslant\ell-1})^2\equiv h_F^2R_F \qquad (\mathrm{mod}\ P). \tag{12}$$

Note that

$$\prod_{j=1}^{\ell-1}(G^{j((q^d-1)/\ell)}-1)\equiv(-1)^{\ell-1}\,\ell \qquad (\mathrm{mod}\ P).$$

Thus, if we combine formulas (8) and (12), we have proved:

THEOREM 2. *Let F be a totally real subfield of the Pth cyclotomic function field. Set $\ell=[F:Q]$, $\ell\geqslant 2$. Let G be a primitive root modulo P. Let h_F be the ideal class number of O_F and let R_F be the regulator modulo P of F. We have*

$$\ell^{2(\ell-2)}h_F^2R_F\equiv(\det(G^{(i-1)\,j((q^d-1)/\ell)})_{1\leqslant i,\,j\leqslant\ell-1})^2 \times\prod_{j=1}^{\ell-1}\frac{B(j((q^d-1)/\ell))^2}{\Gamma^2_{j((q^d-1)/\ell)}} \qquad (\mathrm{mod}\ P).$$

3. KUMMER SUBGROUPS OF UNITS

In this section, we prove Gekeler's conjecture in some special cases.

Again, let F be a totally real subfield of the Pth cyclotomic function field. Let B be a subgroup of E_F, we define the Kummer subgroup of B to be

$$B^{\mathrm{Kum}} = \{ b \in B, \exists \alpha \in O_F, b \equiv \alpha^p \pmod{P} \}.$$

Note that $(B)^p \subset B^{\mathrm{Kum}}$.

For all $a \in Z$, $a \not\equiv 0 \pmod{P}$, let $\sigma_a \in \mathrm{Gal}(Q(\lambda_P)/Q)$ such that $\sigma_a(\lambda_P) = [a]_C(\lambda_P)$. Then we have an isomorphism of groups: $(Z/PZ)^* \to \mathrm{Gal}(Q(\lambda_P)/Q)$, $\bar{a} \mapsto \sigma_a$. Let ω be the Teichmüller character modulo P, i.e., ω: $\mathrm{Gal}(Q(\lambda_P)/Q) \to \mathbb{F}_P^*$ is a group homomorphism such that

$$\omega(\sigma_a) \equiv a \qquad \pmod{P}.$$

Set $\chi = \omega^{(q^d-1)/\ell}$, where $[F : Q] = \ell$. Set $G_F = \mathrm{Gal}(F/Q)$. For $j = 1, ..., \ell$, we set

$$e_j = \frac{1}{\# G_F} \sum_{\tau \in G_F} \chi^j(\tau)\, \tau^{-1} \in \mathbb{F}_P[G_F].$$

Now, if A is a $\mathbb{F}_P[G_F]$-module, we set

$$A(j) = e_j A.$$

We have

$$A = \bigoplus_{j=1}^{\ell} A(j).$$

In particular, $\mathscr{E} = (E_F/(Cyc_F(E_F)^p)) \otimes_{\mathbb{F}_p} \mathbb{F}_P$ is a $\mathbb{F}_P[G_F]$-module. Furthermore the Frobenius of $\mathbb{F}_P/\mathbb{F}_p$ acts on this latter module. Therefore, if $j \in \{1, ..., \ell - 1\}$, for $j_1, j_2 \in [j]_F$ we have

$$\mathscr{E}(j_1) \simeq \mathscr{E}(j_2). \tag{13}$$

Therefore we can define $\mathscr{E}([j]_F)$.

David Goss has proved an analogue of Gras's conjectures for function fields [5, Proposition 2.5], in particular we have:

THEOREM 3. *Let C_F be the p-Sylow subgroup of the ideal class group of O_F (see Section 1). Then*

$$C_F([j]_F) \neq (0) \Leftrightarrow \mathscr{E}([j]_F) \neq (0).$$

Now, we are ready to prove:

THEOREM 4. *Let F be a totally real subfield of the Pth cyclotomic function field. Then*:

(i) $(E_F)^p = E_F^{\mathrm{Kum}}$ *if and only if* $R_F \not\equiv 0 \pmod P$;

(ii) *if* $R_F \not\equiv 0 \pmod P$ *then Gekeler's conjecture is true for the field F* (*see Section* 1).

Proof. Let R_F be the regulator modulo P of F. Note that $\mathscr{L}$ induces a morphism of $\mathbb{F}_P[G_F]$-modules: $(E_F/E_F^{\mathrm{Kum}}) \otimes_{\mathbb{F}_p} \mathbb{F}_p \to \bigoplus_{j=1}^{\ell-1} \mathbb{F}_p \lambda_L^{j(q^d-1)/\ell}$, $\sum \varepsilon_i \otimes \zeta_i \mapsto \sum \mathscr{L}(\varepsilon_i)\, \zeta_i$, because $E_F \cap (U_K)^p\, U_K^{(q^d-1)} = E_F^{\mathrm{Kum}}$ (see formulas (3) and (10)). Note also that, for $\tau \in G_F$, we have

$$\tau(\lambda_L^{j(q^d-1)/\ell}) = \chi^j(\tau)\, \lambda_L^{j(q^d-1)/\ell},$$

where we identify G_F and $\mathrm{Gal}(FQ_P/Q_P)$. Thus

$$R_F \not\equiv 0 \pmod P \Leftrightarrow \dim_{\mathbb{F}_P} \mathscr{L}\left(\frac{E_F}{E_F^{\mathrm{Kum}}} \otimes_{\mathbb{F}_p} \mathbb{F}_P\right) = \ell - 1.$$

But

$$\dim_{\mathbb{F}_P} \frac{E_F}{E_F^{\mathrm{Kum}}} \otimes_{\mathbb{F}_p} \mathbb{F}_P = \dim_{\mathbb{F}_p} \frac{E_F}{E_F^{\mathrm{Kum}}}.$$

Thus

$$R_F \not\equiv 0 \pmod P \Leftrightarrow E_F^{\mathrm{Kum}} = (E_F)^p.$$

Now, we suppose that $R_F \not\equiv 0 \pmod P$. Set

$$\mathscr{C} = \frac{Cyc_F}{Cyc_F^{\mathrm{Kum}}} \otimes_{\mathbb{F}_p} \mathbb{F}_P.$$

Then $\mathscr{L}$ induces an injective morphism of $\mathbb{F}_P[G_F]$-modules,

$$\mathscr{C} \to \bigoplus_{j=1}^{\ell-1} \mathbb{F}_P \lambda_L^{j(q^d-1)/\ell}.$$

Recall that, since $R_F \not\equiv 0 \pmod P$, we have

$$\frac{E_F}{E_F^{\mathrm{Kum}}} \otimes_{\mathbb{F}_p} \mathbb{F}_P \simeq \bigoplus_{j=1}^{\ell-1} \mathbb{F}_P \lambda_L^{j(q^d-1)/\ell}.$$

Now, by formulas (8) and (10), we have

$$\mathscr{C}(j) \neq (0) \Leftrightarrow B\left(j\left(\frac{q^d-1}{\ell}\right)\right) \equiv 0 \pmod{P}.$$

Thus, since $E_F^{\mathrm{Kum}} = (E_F)^p$, we get

$$\mathscr{E}(j) \neq (0) \Leftrightarrow B\left(j\left(\frac{q^d-1}{\ell}\right)\right) \equiv 0 \pmod{P}.$$

Thus, by formula (13), we have

$$\mathscr{E}([j]_F) \neq (0) \Leftrightarrow \forall j' \in [j]_F\, B\left(j'\left(\frac{q^d-1}{\ell}\right)\right) \equiv 0 \pmod{P}.$$

Now, apply Theorem 3. ∎

Note that, by [4, Remark 2.10], we do not have in general $R_F \not\equiv 0 \pmod{P}$.

Let's make some remarks. Let F be a totally real subfield of $Q(\lambda_P)$, $[F:Q]=\ell$. Let $\varepsilon \in E_F$, we denote the subgroup of E_F generated by $\mathbb{F}_q^*$ and $\sigma(\varepsilon)$, $\sigma \in G_F = \mathrm{Gal}(F/Q)$, by B_ε. We set

$$R_\varepsilon \equiv (\det(\varphi_{j(q^d-1)/\ell}(\sigma(\varepsilon)))_{1 \leqslant j \leqslant \ell-1,\, \sigma \in G_F \setminus \{1\}})^2 \pmod{P}.$$

It is clear that

$$R_\varepsilon \not\equiv 0 \pmod{P} \Leftrightarrow \prod_{j=1}^{\ell-1} \varphi_{j(q^d-1)/\ell}(\varepsilon) \not\equiv 0 \pmod{P}.$$

Furthermore, if $R_\varepsilon \not\equiv 0 \pmod{P}$, then B_ε is of finite index in E_F, thus by the results of Section 2,

$$R_F \not\equiv 0 \pmod{P}.$$

Now, we show how to construct fields F such that $R_F \not\equiv 0 \pmod{P}$. Assume that $q \geqslant 3$. Let $\ell \geqslant 2$ be an integer such that $q \equiv 1 \pmod{\ell}$. Let $A \in Z$, $\deg A \geqslant 1$, A unitary and let $a \in \mathbb{F}_q^*$ such that $P = A^\ell - a$ is irreducible in Z. For example, let a in $\mathbb{F}_q^*$ such that $\mathbb{F}_q^* = \langle a \rangle$ and let $n \geqslant 0$ be an integer (we assume that $q \equiv 1 \pmod 4$ if $\ell^{n+1} \equiv 0 \pmod 4$), by [6, Theorem 3.75], $(T^{\ell^n})^\ell - a$ is irreducible in Z.

Note that, since $d = \deg P \equiv 0 \pmod{\ell}$, there exists $\mu \in \mathbb{F}_P^*$ such that

$$\mu^\ell = 1.$$

Set

$$\theta = \mu\lambda_L^{(q^d-1)/\ell} \in K.$$

Then

$$\theta^\ell = P.$$

Set $F = Q(\theta) \subset K$. Then, by class field theory, F is contained in the maximal real subfield of $Q(\lambda_P)$. Let $\varepsilon = A - \theta$, note that $A \not\equiv 0 \pmod P$. We have

$$N_{F/Q}(\varepsilon) = a \in \mathbb{F}_q^*.$$

Thus $\varepsilon \in E_F$. Set

$$h(X) = A - \mu X^{(q^d-1)/\ell}.$$

Then

$$\frac{h'(\lambda_L)}{h(\lambda_L)} \equiv \sum_{n=1}^{\ell-1} \frac{\mu^n}{A^n \ell} \lambda_L^{(n(q^d-1)/\ell)-1} \qquad \pmod{\mathfrak{p}_K^{q^d-2}}.$$

Thus, for $1 \leqslant j \leqslant \ell - 1$, we have

$$\varphi_{j(q^d-1)/\ell}(\varepsilon) \equiv \frac{\mu^j}{A^j \ell} \qquad \pmod P.$$

Therefore $R_\varepsilon \not\equiv 0 \pmod P$, thus $R_F \not\equiv 0 \pmod P$.

Example. Let p be a prime number, $p \equiv 2 \pmod 3$. Let q be an even power of p. Let a in $\mathbb{F}_q^* \backslash (\mathbb{F}_q^*)^3$. Set $P = T^3 - a \in Z = \mathbb{F}_q[T]$. Let $F = Q(\sqrt[3]{P})$. Then, by our previous results,

$$h_F \equiv 0 \pmod p \Leftrightarrow B\left(\frac{q^3-1}{3}\right) \equiv 0 \qquad \pmod P.$$

Now, by [4, Theorem 4.1] and by direct computation, we have that $h_F \equiv 0 \pmod p$ if and only if we have the following two congruences modulo p:

(i) $\sum_{i=0,\, i\equiv 0(3)}^{(q-1)/3} (-1)^i C^i_{((q-1)/3)+i} \equiv \sum_{i=0,\, i\equiv 1(3)}^{(q-1)/3} (-1)^i C^i_{((q-1)/3)+i} \pmod p$,

(ii) $\sum_{i=0,\, i\equiv 1(3)}^{(q-1)/3} (-1)^i C^i_{((q-1)/3)+i} \equiv \sum_{i=0,\, i\equiv 2(3)}^{(q-1)/3} (-1)^i C^i_{((q-1)/3)+i} \pmod p$, where $C_n^k = n!/(k!(n-k)!)$.

Note that, in the case of number fields, we can also define a regulator modulo p. It would be a consequence of the Kummer–Vandiver conjecture that this latter incongruent to zero modulo p if and only if p is regular (see [2] for more details).

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